Perturbative Analysis of the Interacting Rotating Bose Gas

By
Schuyler MOSS

Senior Honors Thesis
Department of Physics and Astronomy
University of North Carolina at Chapel Hill

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Approved:

Dr. Joaquín DRUT, Thesis Advisor
Dr. Lu-Chang QIN, Reader
Dr. Gökçe BAŞAR, Reader

Honor Pledge

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- No unauthorized assistance has been received or given in the completion of this work.

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UNIVERSITY OF NORTH CAROLINA AT CHAPEL HILL

Abstract

Joaquín Drut
Department of Physics and Astronomy

Bachelor of Science with Honors

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by Schuyler MOSS

The study of ultracold quantum systems has expanded dramatically since the first experimental realization of Bose-Einstein Condensation in atomic alkali gases in 1995 [1]. Furthermore, there is particular interest in the study of rotating condensates because of the formation of quantized vortices in a triangular lattice. This strange behavior is a consequence of the fact that interacting ultracold quantum systems enter the superfluid phase, which can only be understood through the consideration of inter-particle interactions. This thesis performs a first-order approximation for the expectation value of the angular momentum for a system of interacting, rotating Bose gas in order to characterize the effects of these interactions on the angular momentum of the system. Using perturbative expansion, this calculation can be performed analytically, and the results can be used as a base-line estimation for fully numerical Monte Carlo approaches. Additionally, the results presented in this thesis can be used to better understand how properties of the system, such as the moment of inertia, change as a function of the rotational frequency.

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Chapter 1

Introduction

1.1 Ultracold Quantum Gases

Particles in nature come in two distinct families: bosons and fermions. Systems of many bosons are paradigmatic in that they feature Bose-Einstein condensation (BEC), which is a phenomenon predicted by quantum statistical mechanics, even in the absence of interactions [2]. A gas will undergo BEC when the system is cooled below a critical temperature, at which point a large portion of the atoms in the system accumulate in the ground state [3]. Because the gas must be sufficiently cooled before it undergoes condensation, it was not until the development of laser cooling and magnetic trapping that atoms were able to be cooled enough or brought to an appropriate density so that BEC could be observed [2],[4].

After the initial experimental observation of BEC, its properties were studied for many different realizations, namely bosonic gases for different alkali metals. Eventually, it was discovered that fermionic gases also experience this phenomenon due to the formation of Cooper pairs [4]. When two fermions experience attractive interactions, in the presence of a Fermi sea, they form a bound state (the Cooper pair) that behaves as a boson would [5]. Because Cooper-pair formation requires attractive interactions, non-interacting Fermi systems do not display the Bose-Einstein Condensation of pairs. This discovery, and its relation to the understanding of superfluidity and superconductivity, illuminated the importance of the inter-particle interactions within systems of ultracold gases which lead to more in depth studies of these interactions for the cases of Bosons and Fermions.

With this new perspective of the interactions in an ultracold system, researchers were able to establish a connection between these interactions and the superfluid phase of the Bose-Einstein Condensates. When ultracold gases undergo BEC they can enter the superfluid phase, which is a fluid with zero shear viscosity. Superfluidity can only be understood when interactions are taken into consideration, even for a system of ideal, noninteracting Bose gas for which BEC still occurs [6]. One of the notable discoveries during this period of exploration, and a direct consequence of superfluidity, was the realization of the quantized vortices that appear when a system of condensed, superfluid gas is forced to rotate (in the case of ultracold gases it is stirred with lasers) [4]. This paper will focus on such an interacting, rotating system of Bosons. The rotation of condensates will be discussed further in the following section.

The realization of a Bose-Einstein condensate in 1995 [1] marked a turning point in the history of the experimental study of ultracold quantum gases. Since then, increasingly sophisticated techniques have enabled experimentalists to study a wide range of

systems (both bosonic and fermionic) and measure a long list of properties. Currently, ultracold atomic gases represent by far the cleanest and most malleable systems in quantum many-body physics. They have become important for a variety of fields ranging from quantum computing to high-energy, condensed matter, and nuclear physics, due to their ability to mimic strongly interacting matter in extreme regimes of density and temperature which are otherwise not easily available (if at all).

1.2 Rotating Condensates

In addition the fact that angular momentum is a conserved quantity that allows us to probe a system by coupling and angular velocity source to it, these rotating systems are important to study because they exhibit unique properties and behavior that have proven to have applications in many fields of physics [7]. One such behavior is how these condensates respond to rotation. Because of their superfluid phase, they do not experience rigid body rotation as a normal fluid would, but rather, a superfluid condensate will form an array of quantized vortices when rotated, as mentioned previously [8]. In fact, the hallmark of a superfluid condensate, as opposed to a mere condensate (which may happen in the absence of interactions) is the formation of such an array of vortices [7].

To understand the formation of vortices, we will follow closely the work of Cooper and Fetter ([8] and [9], respectively). We start by looking at the macroscopic condensate wave function of the system:

$$\Psi(\mathbf{r},t) = |\Psi(\mathbf{r},t)| \exp[i\Phi(\mathbf{r},t)] \tag{1.1}$$

which is written in terms of the magnitude, $|\Psi|$, and the phase, Φ. The condensate particle density of this system, $n(\mathbf{r},t)$, is equal to the square magnitude of this wave function because of the fact that this wave function is normalized in such a way as to satisfy

$$\int dV |\Psi(\mathbf{r})|^2 = N_0 \approx N, \tag{1.2}$$

where N_0 is the number of particles in the system that are in the condensate and N is the total number of particles in the system. The approximation in this equation is due to the fact that the system of interest is a dilute, low-temperature gas and most of the particles can be found in the condensate. The particle current density takes the following form

$$\mathbf{j} = \frac{\hbar}{M}(\Psi^* \nabla\Psi - \Psi \nabla\Psi^*) = |\Psi|^2 \frac{\hbar}{M} \nabla\Phi, \tag{1.3}$$

where M is the mass of the system. Recalling the relation between $\mathbf{j}$ and $\mathbf{v}$ ($\mathbf{j} = n\mathbf{v}$) and recognizing $|\Psi|^2$ as n, the velocity can be identified as

$$\mathbf{v}(\mathbf{r},t) = \frac{\hbar}{M} \nabla\Phi. \tag{1.4}$$

The circulation, κ, of a fluid at a point in time is defined in terms of a line integral of the velocity around a closed path:

$$\kappa = \oint_{\mathcal{C}} d\mathbf{l} \cdot \mathbf{v} = \frac{\hbar}{M} \oint_{\mathcal{C}} d\mathbf{l} \cdot \nabla \Phi \tag{1.5}$$

Because wave functions must be single valued, this quantity must be an integer multiple of 2π, meaning the circulation, κ, is an integer multiple of $2\pi\hbar/M$.

The vorticity of the entire system, which is the curl of the velocity, vanishes since the velocity contains a gradient, but as shown above, there is a non-zero circulation around the quantized vortex lines. This is because the phase, Φ, could have singularities around which the phase changes by an integer multiple of 2π. These singularities are line-like in the 3D case or point-like in the 2D case and are the quantized vortex lines (or points) around which there are is non-zero circulation [8]. This can be interpreted as the system pushing the rotation into tiny regions of space in order that the total circulation of the system, which is the vorticity, will vanish. These mathematical results, which demonstrate the formation of the quantized vortices that have been experimentally observed, are a consequence of the fact that the system can be represented with the macroscopic wave function written in Eq. (1.1), which is only permissible because of the fact that a large number of the particles in the system are occupying the same state.

Although there is a plentiful body of work that focuses on the formation of these vortices and the lattices in which they arrange themselves, there is still much to learn about how these systems behave in more extreme regimes of temperature, pressure, or angular frequency. For that reason, the continued study of rotating condensates, in any capacity, can be seen as a stepping stone to a more comprehensive understanding of these systems.

Chapter 2

Background Theory

2.1 Quantum Statistical Mechanics

Quantum statistical mechanics is the application of statistical mechanics to quantum mechanical systems. It makes use of statistical methods and probability theory to examine thermodynamic properties of a system in equilibrium and relate them to the underlying microscopic properties. The partition function is an essential concept in quantum statistics as it connects the Hamiltonian (or Lagrangian) to equilibrium thermodynamics. In general, the term "partition function" means something different depending on the context or field in which it is being discussed. This section serves to outline the meaning of the partition function in the context of Quantum Statistical Mechanics and to derive the partition function for a system consisting of noninteracting bosons. The path-integral formulation of quantum field theory will be introduced as an alternative way to examine the partition function for a system of interacting bosons. While we will highlight the unity between the two forms of the partition function, a proper derivation of the connection between the two formalisms is beyond the scope of this work.

2.1.1 Partition Functions

In statistical mechanics, the partition function describes the statistical properties of a system in thermodynamic equilibrium. The general form of the partition function is

$$Z = \sum_{\alpha} e^{-\beta E_{\alpha}}, \tag{2.1}$$

where α represents a set of quantum numbers that define the states over which the expression is summed.

The partition function of a system depends on the properties (namely, the energies) of the microstates occupied by the particles in the system. The probabilistic behavior of systems in thermodynamic equilibrium is a direct result of the fact that systems are made up of particles that exist in these microstates. This means that the partition function, which stores all of the information about the microstates of a system, is the key that unlocks the thermodynamic properties of a system such as the total energy, free energy, entropy, and pressure, amongst other things.

Derivation of the Partition Function for Free Bosons

Bosons are subatomic particles such as photons or gluons that have integer spin values. More than one boson can occupy the same single-particle state, which is the main distinction between bosons and fermions.

To begin the derivation of the bosonic partition function, we consider a system of noninteracting bosons. Each individual boson can occupy any microstate, each with a corresponding energy, ϵ_i. If n_{qi} represents the number of bosons in a particular microstate (i) for a given macrostate (q) then there are N_q total bosons in each macrostate of the system and the corresponding total energy is E_q, where

$$N_q = \sum_i n_{qi} \qquad E_q = \sum_i \epsilon_i n_{qi}. \tag{2.2}$$

The Grand Canonical partition function of such a system can be calculated by summing over all possible microscopic configurations of the system, allowing both the energy and particle number to fluctuate and introducing parameters β and μ, respectively the inverse temperature and the chemical potential, to fix the average energy and particle number. Thus,

$$\mathcal{Z} = \sum_q \exp[-\beta(E_q - \mu N_q)]. \tag{2.3}$$

Since E_q and N_q can be expressed in terms of the bosonic microstates, the grand partition function can be written in terms of those microstates as follows:

$$\mathcal{Z} = \sum_q \exp[-\beta \sum_i n_{qi}(\epsilon_{qi} - \mu)] = \sum_q \prod_i \exp[-\beta n_{qi}(\epsilon_{qi} - \mu)]. \tag{2.4}$$

Summing over all of the macrostates of a system is equivalent to summing over all possible values for the number of bosons occupying each microstate. This is because there is a macrostate in which there are zero bosons in each of the microstates, many macrostates in which there is one boson in one of the microstates, and many more macrostates in which there are many bosons in many of the microstates [10]. Therefore the partition function becomes

$$\mathcal{Z} = \prod_j \sum_{n_j=0}^{\infty} \exp[-\beta n_j(\epsilon_j - \mu)]. \tag{2.5}$$

Each of the sums over the various n_i's go from zero to infinity, so the indices are no longer necessary, except on the energies that correspond to specific microstates. Therefore the partition simplifies to

$$\mathcal{Z} = \prod_i \sum_{n=0}^{\infty} \exp[-\beta n(\epsilon_i - \mu)]. \tag{2.6}$$

This can be simplified even further, using the fact that $1 + x + x^2 + ... = \frac{1}{1-x}$. Therefore the final form of the partition function for free bosons can be written as follows:

$$\mathcal{Z} = \prod_i [1 - \exp[-\beta(\epsilon_i - \mu)]]^{-1}. \tag{2.7}$$

It is important to acknowledge the fact that this is a partition function for *free* bosons and that the system of interest for this project is ultimately an *interacting* rotating Bose gas, i.e. there will be interactions between the bosons, meaning Eq. (2.7) will no longer apply. Moreover, a derivation of the partition function for such an interacting system would be very complex if it were done using the same formalism used in the derivation above.

To address the interacting case, the path integral formulation of quantum field theory is much better suited, and as a result of the well-developed mathematical and computational techniques that exist around path integrals, it will be the formulation of choice for the remainder of this paper.

2.1.2 Interacting Bosons and a Path Integral Formulation of Quantum Field Theory

In quantum field theory the partition function is often defined as follows:

$$Z = \int \mathcal{D}\varphi e^{-S[\varphi]}. \tag{2.8}$$

It takes the form of a functional integral, which is an integration over a space of functions, $\mathcal{D}\varphi$, rather than over a region of space. In this equation, $S[\varphi]$ is the action functional, which will be specified below for our case of interest.

As in quantum statistical mechanics, the partition function plays a central role in quantum field theory. In the latter, it is often used as the generating functional for all correlation functions of a system. A correlation function is a functional expectation value of the product of field operators that are evaluated at different positions. In other words, the partition function (suitably modified to include sources, see Eq. (2.15)) can be used to determine the correlation between physical properties of a system at different spatial positions in the system, shedding light on the role of interactions in inducing correlations at varying length scales. Thus, the partition function in quantum field theory, much like the partition function in quantum statistical mechanics, stores information about the physical properties and interactions of a system.

Although Eq. (2.7) and Eq. (2.8) do not appear to have much in common, especially when considering the fact that the particular system considered in this paper is one with a complex boson field and an action of the form shown in Eq. (2.9), there is a distinct similarity between Eq. (2.1) and Eq. (2.8). In Eq. (2.1), the sum is over all possible microstates of the system. On the other hand, in Eq. (2.8), φ is a function of space and time, which represents any possible configuration of the system. The action that appears in Eq. (2.8) depends on the particular shape of each configuration, much like the energies in Eq. (2.1) depend on the quantum numbers. Because an integral can be interpreted as a continuous sum, all of these similarities reflect the idea that these two equations are, indeed, formally equivalent [11].

Although a formal derivation relating Eq. (2.1) and (2.8) will not be given, it is worth discussing a method one might use to perform this derivation. Equation (2.8) is a "Wick-rotated" version of the partition function one obtains by applying the principle of least action to quantum mechanical systems [11],[12]. Wick-rotation is when a real-valued variable is replaced with an imaginary variable. In the case of partition functions, time

gets replaced by imaginary time. This wick-rotation is central to the relationship between the [12] forms of the partition function in quantum statistical mechanics and quantum field theory.

The specific action for our system of interest depends on a complex boson field ($\varphi(\mathbf{x},\tau) = \varphi^1(\mathbf{x},\tau) + i\varphi^2(\mathbf{x},\tau)$) living in three spatial dimensions. Previous work by Hayata and Yamamoto [13] considered the rotating version of this system by placing it in a rotating frame. The action then takes the following form:

$$\begin{aligned} S = \int d\tau d^3x [\varphi^*(\partial_\tau - \mu)\varphi + \frac{1}{2m}|(\nabla - im\mathbf{\Omega}\times\mathbf{x})\varphi|^2 \\ -\frac{1}{2}(x^2+y^2)\Omega^2|\varphi|^2 + \frac{1}{4}\lambda|\varphi|^4], \end{aligned} \tag{2.9}$$

where μ is the chemical potential, m is the mass of the boson, Ω is the rotational frequency, and $\mathbf{\Omega} = \Omega\hat{z}$, so that the rotation is about the z-axis. Finally, λ is the coupling constant that controls the strength of the self-interaction. By expanding the kinetic term we obtain

$$\begin{aligned} \frac{1}{2m}|(\nabla - im\mathbf{\Omega}\times\mathbf{x})\varphi|^2 = \varphi^*\frac{\nabla^2}{2m}\varphi - \frac{i}{2}\varphi^*\left(\nabla\cdot(\mathbf{\Omega}\times\mathbf{x}) - (\mathbf{\Omega}\times\mathbf{x})\cdot\nabla\right)\varphi \\ +\frac{1}{2}(x^2+y^2)\Omega^2|\varphi|^2, \end{aligned} \tag{2.10}$$

where $\mathbf{x} = (x,y,z)$. We thus see, explicitly, how the centrifugal term cancels out against the last term of the above equation. The remaining terms that depend on Ω represent the coupling to angular momentum, which is what we are after. Below, we will continue to use the above formulation in order to elucidate the lattice form of the angular momentum coupling, which will turn out to require link variables. Once that is accomplished, we will proceed assuming that the Ω^2 term has been cancelled out, as if we were operating in an inertial frame in which the system is coupled to an angular momentum source (i.e. a stirring spoon), rather than use the non-inertial version (Eq. (2.9)).

Finally, it is important to keep in mind that systems that rotate about an axis must somehow be contained by an external trapping potential, otherwise the fastest particles at the edge of the system would ultimately escape. This can be done by way of boundary conditions, or by adding such a potential, which would simply add a new term to the action of the form

$$+\frac{1}{2}(x^2+y^2)\Omega_{\text{tr}}^2|\varphi|^2, \tag{2.11}$$

where Ω_{tr} is the trapping potential frequency. To prevent a rotating system from escaping such a trap, one must have $\Omega_{\text{tr}} > \Omega$.

2.2 Gaussian Expectation Values

The natural starting point for analyzing quantum field theories is the noninteracting case. In the absence of interactions, the action of a system is quadratic in the fields and therefore the integrand in the partition function takes on a simple Gaussian form. When interactions are turned on, the mathematics of Gaussian integrals still plays a crucial role, as we will show below, and is one of the central tools in Quantum Field

Theory. In this section, many of the mathematical equations used in the main calculation of this thesis (Chapter 3) are derived and discussed. This section synthesizes much of the information presented in Chapter 2 of Jean Zinn-Justin's *Phase Transitions and Renormalization Group* [14].

2.2.1 Gaussian Integrals and their Solutions

The simplest Gaussian integral takes the well known form

$$Z(a) = \int dx e^{-ax^2} = \sqrt{\frac{2\pi}{a}} \tag{2.12}$$

which corresponds to a zero-dimensional quantum field theory.

To take a step towards a full-fledged field theory, consider the following multidimensional Gaussian integral

$$Z(\mathbf{A}) = \int d^n x e^{-A(\mathbf{x})} = (2\pi)^{n/2} \det(\mathbf{A})^{-1/2}, \tag{2.13}$$

where

$$A(\mathbf{x}) = \frac{1}{2} \sum_{i,j=1}^{n} x_i A_{i,j} x_j. \tag{2.14}$$

In later chapters, there will be as many integrals n as points in spacetime, and the variables x_i will become fields $\varphi(\mathbf{x})$.

Next, consider a more general Gaussian integral

$$Z(\mathbf{A}, \mathbf{b}) = \int d^n x e^{-A(\mathbf{x}) + \mathbf{b}\cdot\mathbf{x}}, \tag{2.15}$$

where $A(\mathbf{x})$ takes the same form (Eq. (2.14)), and $\mathbf{b} \cdot \mathbf{x}$ is an inner product that is defined as

$$\mathbf{b} \cdot \mathbf{x} = \sum_{i=1}^{n} b_i x_i. \tag{2.16}$$

By completing the square in the exponent one can reduce Eq. (2.15) to an equation that takes the form of Eq. (2.13). The solution to the integral can then be easily found:

$$Z(\mathbf{A}, \mathbf{b}) = (2\pi)^{n/2} \det(\mathbf{A})^{-1/2} e^{\mathbf{\Delta}(\mathbf{b})}, \tag{2.17}$$

where

$$\mathbf{\Delta}(\mathbf{b}) = \frac{1}{2} \sum_{i,j=1}^{n} b_i \Delta_{i,j} b_j \quad \text{where } \Delta_{i,j} = A^{-1}_{i,j}. \tag{2.18}$$

This calculation is done in more detail and with complex variables in Appendix A.

The above integral, in which b_i is now a free variable, will be essential when generalized to its full quantum field theory form. There, the b_i will play the role of external sources. By differentiating the partition function with respect to those sources, we will generate the expectation values we want to study. In the next section we take steps toward making contact with those concepts.

2.2.2 Expectation Values and Wick's Theorem

Let us define the expectation value of $F(\mathbf{x})$, a function of the vector variable $\mathbf{x}$, as

$$\langle F(\mathbf{x})\rangle = \mathcal{N}(\mathbf{A})\int d^n x F(\mathbf{x}) e^{-A(\mathbf{x})}, \tag{2.19}$$

where $\mathcal{N}(\mathbf{A})$ is a normalization constant that ensures $\langle 1\rangle = 1$. Setting $F(\mathbf{x}) = 1$, then $\langle F(\mathbf{x})\rangle = \mathcal{N}(\mathbf{A})Z(\mathbf{A},0)$ meaning the normalization constant, $\mathcal{N}(\mathbf{A})$, must equal $Z(\mathbf{A},0)^{-1}$.

In the following, we will use the expression

$$\frac{Z(\mathbf{A},\mathbf{b})}{Z(\mathbf{A},0)} = \frac{1}{Z(\mathbf{A},0)}\int d^n x e^{-A(\mathbf{X})+\mathbf{b}\cdot\mathbf{x}} = \frac{1}{Z(\mathbf{A},0)}\int d^n x e^{-A(\mathbf{X})} e^{\mathbf{b}\cdot\mathbf{x}} = \langle e^{\mathbf{b}\cdot\mathbf{x}}\rangle. \tag{2.20}$$

This function, $\langle e^{\mathbf{b}\cdot\mathbf{x}}\rangle$, is a generating function of the expectation values of the monomials. Indeed, taking derivatives of $\langle e^{\mathbf{b}\cdot\mathbf{x}}\rangle$ with respect to b_i brings down powers of x_i, such that, in the limit $\mathbf{b} = 0$, the only non-vanishing terms are those powers of x_i. The equations below show how this information can be used to calculate these expectation values. Taking derivatives on both sides of Eq. (2.20) and taking the limit $\mathbf{b} = 0$ one obtains:

$$\frac{\partial}{\partial b_{i_1}}\frac{\partial}{\partial b_{i_2}}\cdots\frac{\partial}{\partial b_{i_l}}\langle e^{\mathbf{b}\cdot\mathbf{x}}\rangle\Big|_{\mathbf{b}=0} = \frac{\partial}{\partial b_{i_1}}\frac{\partial}{\partial b_{i_2}}\cdots\frac{\partial}{\partial b_{i_l}}\frac{Z(\mathbf{A},\mathbf{b})}{Z(\mathbf{A},0)}\Big|_{\mathbf{b}=0}. \tag{2.21}$$

Recalling Eq. 2.13 and Eq. 2.15, this becomes

$$\langle x_{i_1}x_{i_2}\ldots x_{i_l}\rangle = \frac{\partial}{\partial b_{i_1}}\frac{\partial}{\partial b_{i_2}}\cdots\frac{\partial}{\partial b_{i_l}}\frac{(2\pi)^{n/2}\det(\mathbf{A})^{-1/2}e^{\mathbf{\Delta}(\mathbf{b})}}{(2\pi)^{n/2}\det(\mathbf{A})^{-1/2}}\Big|_{\mathbf{b}=0}, \tag{2.22}$$

which reduces to

$$\langle x_{i_1}x_{i_2}\ldots x_{i_l}\rangle = \frac{\partial}{\partial b_{i_1}}\frac{\partial}{\partial b_{i_2}}\cdots\frac{\partial}{\partial b_{i_l}}e^{\mathbf{\Delta}(\mathbf{b})}\Big|_{\mathbf{b}=0}. \tag{2.23}$$

Generalizing this further, one can show that if $F(\mathbf{x})$ has a series expansion in $\mathbf{x}$, then its expectation value can be calculated in the following way:

$$\langle F(\mathbf{x})\rangle = \left[F\Big(\frac{\partial}{\partial\mathbf{b}}\Big)e^{\mathbf{\Delta}(\mathbf{b})}\right]\Big|_{\mathbf{b}=0}. \tag{2.24}$$

To understand the importance of these findings, consider what happens when a derivative acts on $e^{\mathbf{\Delta}(\mathbf{b})}$:

$$\frac{\partial}{\partial b_i}e^{\mathbf{\Delta}(\mathbf{b})} = \sum_{j=1}^{n}\Delta_{i,j}b_j e^{\mathbf{\Delta}(\mathbf{b})}. \tag{2.25}$$

This term will vanish in the limit $\mathbf{b} = 0$ because the derivative brought down a factor of b from the exponent. In order for there to be non-zero terms, there must be an even number of derivatives. This means that only the expectation values of even-powered monomials are non-zero, and furthermore the terms that appear in the result will correspond to all possible ways in which differentiating with respect to pairs of

variables b_i, b_j will act on $\Delta(\mathbf{b})$. The result is beautifully encoded in *Wick's theorem*, which we outline next.

Wick's theorem tackles the calculation of the general expectation value $\langle x_{i_1} x_{i_2} ... x_{i_l} \rangle$, where l can be assumed to be an even number because of the previous discussion about how the expectation values of odd monomials are zero. To that end, consider first the possible ways the indices $i_1, i_2, ..., i_l$ can be paired. Let P represent the sets of possible pairings of these indices. For each set of pairings, there are $l/2$ individual pairs of indices, i_p and i_q, and for each pair of indices, there is a corresponding element of $\mathbf{\Delta} = \mathbf{A}^{-1}$, Δ_{i_p,i_q}. Wick's theorem asserts that

$$\langle x_{i_1} x_{i_2} ... x_{i_l} \rangle = \sum_P \Delta_{P_{1_1},P_{1_2}} \Delta_{P_{2_1},P_{2_2}} ... \Delta_{P_{l/2_1},P_{l/2_2}}. \tag{2.26}$$

For example, if the possible i's are 1, 2, 3, 4, the set of all possible pairings is (1 and 2, 3 and 4) or (1 and 3, 2 and 4) or (1 and 4, 2 and 3). Furthermore, Eq. (2.26) can be used to determine $\langle x_1 x_2 x_3 x_4 \rangle$, as shown below:

$$\langle x_1\, x_2\, x_3\, x_4 \rangle = \Delta_{1,2}\Delta_{3,4} + \Delta_{1,3}\Delta_{2,4} + \Delta_{1,4}\Delta_{2,3}. \tag{2.27}$$

The assertion of Wick's Theorem is slightly different when considering complex variables. To see this, consider the form of $\mathbf{\Delta}(\mathbf{b}^*, \mathbf{b})$ derived in Appendix A:

$$\mathbf{\Delta}(\mathbf{b}^*, \mathbf{b}) = \mathbf{b}^* \cdot \mathbf{\Delta} \cdot \mathbf{b} = \sum_{i,j=1}^{n} b_i^* \Delta_{i,j} b_j. \tag{2.28}$$

A derivative with respect to b_i^* brings down a factor of b_j from the exponent. As a result, for the complex case, derivatives must come in pairs of derivatives with respect to b_i^* and b_j, rather than simply in even powers as is true for the case with real variables. One can still take a combinatoric approach that is described by Wick's Theorem where the only difference is that the sets of pairings are obtained by considering all the different ways that the indices on derivatives with respect to $\mathbf{b}^*$ can be paired with an index on a derivative with respect to $\mathbf{b}$.

In the calculations performed in chapter 3, the full calculations are carried out using Eq. (2.24); however, Wick's theorem can be used to confirm the results. When performing more complex calculations than the ones shown in this paper, calculations using Eq. (2.24) become tedious, so Wick's theorem is necessary for obtaining results.

2.2.3 Perturbed Gaussian Integrals

When turning on interactions, our integrals will not fit the form of Eq. (2.13), but rather a "perturbed" version. For example, consider the following equation:

$$Z(\mathbf{A}, \lambda) = \int d^n x e^{-A(\mathbf{x},\lambda)} \tag{2.29}$$

This integral appears to take the same form as Eq. (2.13), however $A(\mathbf{x}, \lambda)$ is defined as

$$A(\mathbf{x}, \lambda) = A(\mathbf{x}) + \lambda V(\mathbf{x}) \tag{2.30}$$

with $A(\mathbf{x})$ defined by Eq. (2.14) and $V(\mathbf{x})$ is a general, non-quadratic function (e.g. a cubic or higher-order polynomial). If λ is small, then $\lambda V(\mathbf{x})$ represents a perturbation to a true Gaussian form. Perturbation theory, discussed in more detail in the next section, is one way to calculate integrals that involve this type of "non-Gaussian" integrand.

2.3 Perturbation Theory

Perturbation theory is a method for approximating complicated quantum systems in terms of simpler ones. For example, Eq. (2.9) is the action for an interacting system, where λ is the interaction coefficient. When $\lambda = 0$, the interactions are "turned off" and the the remaining terms represent the action for a noninteracting, but otherwise equivalent system. For the noninteracting case, the action is quadratic in φ, meaning the partition function Eq. (2.8) takes the form of a Gaussian integral. These types of integrals are well studied and can be solved analytically, using the tools of the previous section. In order to calculate the properties of these more interesting, interacting systems, one can simply separate the action into two components, the free part, S_0, and the interaction, V:

$$S[\varphi^*, \varphi] = S_0[\varphi^*, \varphi] + \lambda V[\varphi^*, \varphi]. \tag{2.31}$$

When evaluating the partition function of an interacting system, one can expand the integrand of Eq. (2.8) in a power series in λ, which yields an infinite sequence of Gaussian integrals which can also be solved, order by order in λ. Methods for solving such integrals will be discussed in further detail in the next chapter; however, it is worth highlighting the fact that the expansion of the integrand in powers of λ, which is the central step in perturbation theory, is the trick that allows the resulting integral to be solved analytically. On the other hand, a full fledged non-perturbative treatment of the problem, would require computationally intensive Monte Carlo methods. Because perturbative calculations can be carried out analytically, they are often used as a first estimation and a way to check the results obtained of the more computationally expensive, fully numerical Monte Carlo approaches.

It is important to point out that this paper focuses on a first-order approximation for the quantity we seek, namely the expectation value of the angular momentum for the interacting, rotating Bose gas, meaning we keep only terms that are linear in λ after perturbatively expanding the original calculation. In principle, taking higher powers of λ into account would yield an improved approximation, i.e. one that converges to the true value; however, in practice, this is not always the case. To confound things further, sometimes the approximation improves with higher orders of λ until a particular order is reached, at which point the expansion begins to diverge. Thus, perturbation theory in quantum mechanics and quantum field theory is a much more subtle affair than merely Taylor-expanding. Some of the subtleties that should be considered when examining high-order perturbative expansions go beyond whether or not the expansion converges. For instance, in some cases it is possible for the perturbative series to diverge but it may do so in a way that is summable, which can be determined if one finds the general behavior of the perturbative expansion at high orders. Although these are interesting and important problems to think about, we will not consider them in this paper.

2.4 Calculating for the Interacting Theory

To pull together the ideas that were introduced in previous sections, this discussion will focus on perturbation theory as it relates to expectation values for a system as defined by interacting quantum field theory.

The interactions of the particles in the system that are considered in Eq. (2.9) represent perturbations to an otherwise quadratic action. This is because Eq. (2.31) is of the form shown in Eq. (2.30). As a result, the partition function for this system takes the form of Eq. (2.29). Furthermore, expectation values that take these interactions into consideration, which are expectation values as defined by interacting quantum field theory, take the following form

$$\langle F[\varphi^*, \varphi] \rangle = \frac{1}{Z} \int \mathcal{D}\varphi^\dagger \mathcal{D}\varphi \; F[\varphi^\dagger, \varphi] \; e^{-S[\varphi^\dagger, \varphi, \lambda]}, \tag{2.32}$$

where Z is the partition function of the interacting system and $S[\varphi^*, \varphi, \lambda]$ is the action of the interacting system as defined by Eq. (2.31).

This is the type of expectation value that this thesis aims to calculate: the interacting expectation value of the angular momentum operator. Using perturbation theory, however, this expectation value can be written in simpler terms, as advocated above, namely in terms of Gaussian path integrals. Both the normalization constant, Z^{-1}, and the action, $S[\varphi^*, \varphi, \lambda]$, can be expanded in powers of λ. In doing this, the partition function of the interacting system can be written in terms of the noninteracting partition function which takes the form of a true Gaussian integral, Eq. (2.13), since all noninteracting actions are quadratic, and the expectation value of the interaction terms as defined by *noninteracting* theories, Eq. (2.19). Similarly, the action of the interacting system inside the integral can be expanded in such a way that the interacting expectation value can be written in terms of noninteracting expectation values which take the form of Eq. (2.19). This will become more clear in the calculations that are presented in the following chapter, however the distinction between interacting and noninteracting expectation values is important. Henceforth, noninteracting expectation values (which take the form of Eq. (2.19)) will carry a subscript of 0 to distinguish them from interacting expectation values.

Chapter 3

Main Calculation

This thesis is concerned with determining the effect of the interactions in a trapped, rotating Bose gas. As interactions alter the mass distribution, we expect the moment of inertia and the angular momentum of the system to change. To understand that effect, the interacting expectation value of the angular momentum operator will be calculated.

3.1 Perturbative Expansion of the Interacting Expectation Value

This calculation will take the following form

$$\langle \hat{L} \rangle = \frac{1}{Z} \int \mathcal{D}\varphi^* \mathcal{D}\varphi \hat{L}[\varphi^*, \varphi] e^{-S[\varphi^*, \varphi]}. \tag{3.1}$$

The equation above takes the form of Eq. (2.32) where $F[\varphi^*, \varphi] = \hat{L}[\varphi^*, \varphi]$.

As defined in Chapter 2, the partition function for a system is defined by Eq. (2.8). Generalizing this equation to the complex case, the partition function takes the form

$$Z = \int \mathcal{D}\varphi^* \mathcal{D}\varphi e^{-S[\varphi^*, \varphi]}. \tag{3.2}$$

The system of interest is an interacting one, meaning the action can be split into two components: the interacting and the noninteracting pieces. In Eq. (3.3) below, S_0 represents the noninteracting terms in the action, while V represents the interacting terms. The interaction coefficient λ is a control parameter for our perturbative expansion. The action can therefore be written as follows:

$$S[\varphi^*, \varphi] = S_0[\varphi^*, \varphi] + \lambda V[\varphi^*, \varphi]. \tag{3.3}$$

Since the action is really the sum of its two components, the exponential term $e^{-S[\varphi^*, \varphi]}$ can be split into the product of two exponential terms,

$$Z = \int \mathcal{D}\varphi^* \mathcal{D}\varphi e^{-S_0[\varphi^*, \varphi]} e^{-\lambda V[\varphi^*, \varphi]}. \tag{3.4}$$

A noninteracting action always takes a quadratic form, meaning this equation takes the unnormalized form of the Gaussian expectation value shown in Eq. (2.19), where $F[\varphi^*, \varphi] = e^{-\lambda V(\varphi^*, \varphi)}$. Using this information and expanding $e^{-\lambda V(\varphi^*, \varphi)}$ in powers of λ, the expectation value can be written in terms of the noninteracting expectation value of $V[\varphi^*, \varphi]$. This means the interacting partition function Z can be written in terms of

the noninteracting partition function, Z_0, and the noninteracting expectation value of $V[\varphi^*, \varphi]$:

$$Z = Z_0 \langle e^{-\lambda V[\varphi^*,\varphi]} \rangle_0 = Z_0(1 - \lambda \langle V \rangle_0 + ...). \tag{3.5}$$

The action for this system, a complex boson field in a rotating frame, can be found in chapter Chapter 2, Eq. (2.9). The continuous action is often discretized so that lattice methods can be used to compute the integral. The discretized action is

$$\begin{aligned} S_{lat}[\varphi^*, \varphi] = a^3 \sum_{\tau,\mathbf{x}} \Bigg[& \varphi^*_{\tau,\mathbf{x}}(\varphi_{\tau,\mathbf{x}} - e^{\mu a}\varphi_{\tau-a,\mathbf{x}}) \\ & - \frac{1}{2ma} \sum_i (\varphi^*_{\tau,\mathbf{x}+\hat{i}a} u_i^* \varphi_{\tau,\mathbf{x}} + \varphi^*_{\tau,\mathbf{x}} u_i \varphi_{\tau,\mathbf{x}+\hat{i}a} - 2|\varphi_{\tau,\mathbf{x}}|^2) \\ & - \frac{1}{2} ma(x^2 + y^2)\Omega^2 \varphi^*_{\tau,\mathbf{x}} \varphi_{\tau-a,\mathbf{x}} + \frac{1}{4}\frac{\lambda}{a^2} a^3 (\varphi^*_{\tau,\mathbf{x}} \varphi_{\tau-a,\mathbf{x}})^2 \Bigg], \end{aligned} \tag{3.6}$$

with

$$\begin{aligned} u_i &= \exp(-iam(\mathbf{\Omega} \times \mathbf{x})_i) \\ u_i^* &= \exp(+iam(\mathbf{\Omega} \times \mathbf{x})_i), \end{aligned} \tag{3.7}$$

and where μ is the chemical potential, m is the mass of the boson, ∇ contains the spatial derivatives, and $\mathbf{\Omega} = \Omega\hat{z}$ so that the rotation is about the z-axis. It is important to point out that the action for this system, Eq. (2.9), is mathematically equivalent to that of a system of charged bosons in a magnetic field. This means that, in the rotating frame, particles couple to rotation term in the same way that they would couple to a magnetic field ($\mathbf{B}$):

$$q\mathbf{B} = q\boldsymbol{\nabla} \times (\mathbf{\Omega} \times \mathbf{x}) = 2m\Omega\hat{z}. \tag{3.8}$$

For that reason, "link variables" must be used to represent the rotation in the lattice formulation of the problem [13]. These link variables are u_i and u_i^*, which are defined in Eq. (3.7). Usually, link variables are used to represent gauge fields in lattice field theory because they guarantee gauge invariance. While a more detailed discussion of gauge fields is beyond the scope of this work, it is important to point out their central role in particle physics, as photons, W bosons, and gluons are all gauge fields, and the lattice treatment of quantum chromodynamics (i.e. lattice QCD) relies on link variables to represent the strong force.

Splitting the discretized action into the noninteracting component and the interaction term(s), the following equations for S_0 and V are obtained:

$$\begin{aligned} S_0[\varphi^*, \varphi] = a^3 \sum_{\tau,\mathbf{x}} \Bigg[& \varphi^*_{\tau,\mathbf{x}}(\varphi_{\tau,\mathbf{x}} - e^{\mu a}\varphi_{\tau-a,\mathbf{x}}) \\ & - \frac{1}{2ma} \sum_i (\varphi^*_{\tau,\mathbf{x}+\hat{i}a} u_i^* \varphi_{\tau,\mathbf{x}} + \varphi^*_{\tau,\mathbf{x}} u_i \varphi_{\tau,\mathbf{x}+\hat{i}a} - 2|\varphi_{\tau,\mathbf{x}}|^2) \\ & - \frac{1}{2} ma(x^2 + y^2)\Omega^2 \varphi^*_{\tau,\mathbf{x}} \varphi_{\tau-a,\mathbf{x}} \Bigg] \end{aligned} \tag{3.9}$$

and

$$V[\varphi^*, \varphi] = \frac{a^4}{4} \sum_{\tau, \boldsymbol{x}} (\varphi^*_{\tau, \boldsymbol{x}} \varphi_{\tau - a, \boldsymbol{x}})^2. \tag{3.10}$$

Returning to the calculation of $\langle \hat{L} \rangle$, the normalization constant Z can be rewritten according to Eq. (3.5) and the action can be separated into its noninteracting and interacting components,

$$\langle \hat{L} \rangle = \frac{1}{Z_0(1 - \lambda \langle V \rangle_0 + ...)} \int \mathcal{D}\varphi^* \mathcal{D}\varphi \hat{L}[\varphi^*, \varphi] e^{-S_0[\varphi^*, \varphi]} e^{-\lambda V[\varphi^*, \varphi]}. \tag{3.11}$$

This calculation will be a first order perturbative calculation. This means that all terms on the order of λ^2 or higher will be dropped. Technically $1/(1 - \lambda \langle V \rangle)$ has an infinite series expansion in powers of λ, so in order to be consistent, this term can be expanded in the numerator, and then any terms on the order of λ^2 or higher will be dropped. The exponential term with λV in the exponent inside the integral can also be expanded in powers of λ at this point. Again, because this is a first order perturbative calculation, all terms on the order of λ^2 or higher will be dropped. The expectation value then becomes

$$\langle \hat{L} \rangle \approx \frac{(1 + \lambda \langle V \rangle_0)}{Z_0} \int \mathcal{D}\varphi^* \mathcal{D}\varphi \hat{L}[\varphi^*, \varphi] e^{-S_0[\varphi^*, \varphi]} (1 - \lambda V[\varphi^*, \varphi]), \tag{3.12}$$

which can be split into two separate integrals

$$\begin{aligned} \langle \hat{L} \rangle \approx \frac{(1 + \lambda \langle V \rangle_0)}{Z_0} \Bigg[& \int \mathcal{D}\varphi^* \mathcal{D}\varphi \hat{L}[\varphi^*, \varphi] e^{-S_0[\varphi^*, \varphi]} \\ & - \lambda \int \mathcal{D}\varphi^* \mathcal{D}\varphi \hat{L}[\varphi^*, \varphi] V[\varphi^*, \varphi] e^{-S_0[\varphi^*, \varphi]} \Bigg]. \end{aligned} \tag{3.13}$$

Both integrals, with the $1/Z_0$, take the form of noninteracting expectation values of the operators and functions inside the integrand (Eq. (2.19)). Therefore, the equation above can be simplified as follows:

$$\langle \hat{L} \rangle \approx (1 + \lambda \langle V \rangle_0) \big[\langle \hat{L} \rangle_0 - \lambda \langle \hat{L} V \rangle_0 \big]. \tag{3.14}$$

Multiplying the two terms together and keeping only terms that are linear in λ, the following approximation for $\langle \hat{L} \rangle$ is obtained:

$$\langle \hat{L} \rangle \approx \langle \hat{L} \rangle_0 - \lambda \big[\langle \hat{L} V \rangle_0 - \langle V \rangle_0 \langle \hat{L} \rangle_0 \big]. \tag{3.15}$$

The leading term in the equation above is simply the non-interacting expectation value for the angular momentum. The terms in brackets are a first order correction to the non-interacting expectation value. This approximation is consistent with the case where interactions are "turned off" ($\lambda = 0$). The interacting expectation in such a case is really the non-interacting expectation value and the correction disappears, thus $\langle \hat{L} \rangle = \langle \hat{L} \rangle_0$. A more detailed explanation of the terms in the correction will be given in the following section in conjunction with a discussion of Feynman diagrams.

3.1.1 Feynman Diagrams

In 1948, Richard Feynman introduced Feynman diagrams as a bookkeeping method for long quantum electrodynamics equations [15]. These diagrams, among many other applications, can be used as a pictorial representation of perturbation theory. For that reason, the equation above can be discussed in terms of the diagrams that correspond to each term.

The central components of a Feynman diagram are a vertex that corresponds to the interactions and a contribution from the operator of interest, in our case, $\hat{L}$:

FIGURE 3.1: Components of a Feynman diagram that correspond to the interactions ($\hat{V}$) and the angular momentum ($\hat{L}$).

The number of vertices in the diagram is equivalent to the order of the perturbative calculation.

First-order Feynman Diagrams

A first-order diagram will contain one vertex and the contribution of $\hat{L}$. Because since there are only two components, there is only one way that this diagram can be fully connected, that is, have all its components connected to one another. The fully connected diagram is shown in the figure below. This diagram corresponds to the first

FIGURE 3.2: First order connected contributions.

term, $\langle \hat{L}V \rangle_0$, in the correction to the noninteracting expectation value in Eq. 3.15. In terms of the equation, this term is called the connected contribution.

Similarly, there is only one way that the diagram can be disconnected, which is when the components of the diagram are not *fully* connected. This type of diagram is shown below. This diagram corresponds to the second term, $\langle V \rangle_0 \langle \hat{L} \rangle_0$, in the correction to

FIGURE 3.3: First order disconnected contributions.

the noninteracting expectation value in Eq. 3.15 and is referred to as the disconnected contribution.

The leading term in Eq. 3.15 is the non-interacting expectation value for the angular momentum. In a way, this term contains all of the disconnected contributions of the whole system. For that reason, when making a correction due to interactions, the disconnected contributions must be subtracted from the fully connected contributions. This is because the fully connected contributions encode disconnected contributions, and without subtracting them out, we would essentially be double counting these disconnected contributions between $\langle \hat{L} \rangle_0$ and $\langle \hat{L} V \rangle_0$.

Second-order Feynman Diagrams

In a second-order diagram there will be two vertices (corresponding to λ^2) and the contribution of $\hat{L}$. Because of this extra vertex, there are four different ways that the the diagram can be fully connected. These diagrams are shown below.

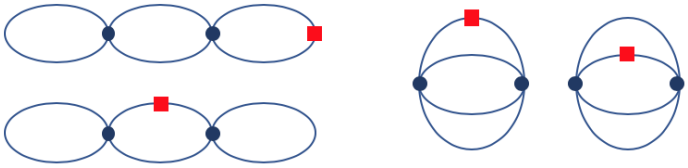

FIGURE 3.4: Second order connected contributions.

There are two ways that the diagram can be disconnected. These diagrams are shown below. It is important to note that the diagram does not have to be fully dis-

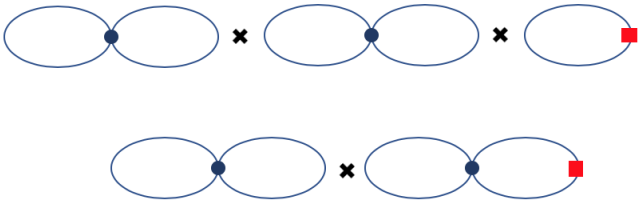

FIGURE 3.5: Second order disconnected contributions.

connected to be considered disconnected. For example, the diagram on the right shows a partially connected diagram where the contribution from $\hat{L}$ connected to one of the vertices. This is considered disconnected and would be subtracted out in a second order calculation because it is essentially a first order connected contribution. When doing a second order calculation, one presumably includes a first order and a second order correction, so there would be a double consideration of first order connected contributions.

The cancellation of connected and disconnected terms occurs at every order of perturbative calculations [14], however further discussion is beyond the scope of this paper.

3.2 Isolating the Angular Momentum Term on the Lattice

In order to begin calculating the terms in Eq. (3.15) above, the angular momentum operator must be isolated from the equation for the discretized action of the system (Eq. (3.6)). To begin, consider the term where the link variables, u_i and u_i^*, appear.

$$-\frac{a^2}{2m}\sum_{\tau,\boldsymbol{x}}\sum_i(\varphi^*_{\tau,\boldsymbol{x}+\hat{i}a}u_i^*\varphi_{\tau,\boldsymbol{x}}+\varphi^*_{\tau,\boldsymbol{x}}u_i\varphi_{\tau,\boldsymbol{x}+\hat{i}a}-2|\varphi_{\tau,\boldsymbol{x}}|^2) \tag{3.16}$$

If there is no rotation, then $\Omega = 0$ and $u_i = u_i^* = 1$. In order to identify the parts of this term which contain the information about the angular momentum of the system, this limit will be added and subtracted from u_i and u_i^*. This will reveal which terms vanish completely in the absence of angular momentum, which are the terms that contain the angular momentum dependence. Making the following substitution,

$$\begin{aligned} u_i^* &\to u_i^* - 1 + 1 \\ u_i &\to u_i - 1 + 1, \end{aligned} \tag{3.17}$$

this term becomes

$$-\frac{a^2}{2m}\sum_{\tau,\boldsymbol{x}}\sum_i(\varphi^*_{\tau,\boldsymbol{x}+\hat{i}a}(u_i^*-1+1)\varphi_{\tau,\boldsymbol{x}}+\varphi^*_{\tau,\boldsymbol{x}}(u_i-1+1)\varphi_{\tau,\boldsymbol{x}+\hat{i}a}-2|\varphi_{\tau,\boldsymbol{x}}|^2). \tag{3.18}$$

Separating the terms that vanish when $\Omega = 0$, from those that do not, this term becomes

$$\begin{aligned} -\frac{a^2}{2m}\sum_{\tau,\boldsymbol{x}}\sum_i(&\varphi^*_{\tau,\boldsymbol{x}+\hat{i}a}(u_i^*-1)\varphi_{\tau,\boldsymbol{x}}+\varphi^*_{\tau,\boldsymbol{x}}(u_i-1)\varphi_{\tau,\boldsymbol{x}+\hat{i}a} \\ &+\varphi^*_{\tau,\boldsymbol{x}+\hat{i}a}\varphi_{\tau,\boldsymbol{x}}+\varphi^*_{\tau,\boldsymbol{x}}\varphi_{\tau,\boldsymbol{x}+\hat{i}a}-2|\varphi_{\tau,\boldsymbol{x}}|^2), \end{aligned} \tag{3.19}$$

where the top line entirely contains the part of the action that is associated with the angular momentum. Although other terms in the action are sensitive to the angular momentum, and might change if $\Omega = 0$, they will not completely vanish the way this term will. From this point on, the angular momentum operator can be identified as

$$\hat{L} = -\frac{a^2}{2m}\sum_{\tau,\boldsymbol{x}}\sum_i(\varphi^*_{\tau,\boldsymbol{x}+\hat{i}a}(u_i^*-1)\varphi_{\tau,\boldsymbol{x}}+\varphi^*_{\tau,\boldsymbol{x}}(u_i-1)\varphi_{\tau,\boldsymbol{x}+\hat{i}a}). \tag{3.20}$$

In terms of derivatives with respect to the sources, the operator can be written as

$$\hat{L} = -\frac{a^2}{2m}\sum_{\tau,\boldsymbol{x}}\sum_i\left((u_i^*-1)\frac{\partial}{\partial b_{\tau,\boldsymbol{x}+\hat{i}a}}\frac{\partial}{\partial b^*_{\tau,\boldsymbol{x}}}+(u_i-1)\frac{\partial}{\partial b_{\tau,\boldsymbol{x}}}\frac{\partial}{\partial b^*_{\tau,\boldsymbol{x}+\hat{i}a}}\right), \tag{3.21}$$

however, Eq. (3.20) will be used in the calculations that follow.

3.3 Calculating the Expectation Value of Angular Momentum on the Lattice

Using Eq. (3.20), the noninteracting expectation value for the angular momentum operator can be written as follows:

$$\begin{aligned}\langle \hat{L} \rangle_0 = -\frac{a^2}{2m} \sum_{\tau,\mathbf{x}} \sum_i \frac{1}{Z_0} \Bigg[\int \mathcal{D}\varphi^* \mathcal{D}\varphi (u_i^* - 1) \varphi^*_{\tau,\mathbf{x}+\hat{i}a} \varphi_{\tau,\mathbf{x}} e^{-S_0[\varphi^*,\varphi]} \\ + \int \mathcal{D}\varphi^* \mathcal{D}\varphi (u_i - 1) \varphi^*_{\tau,\mathbf{x}} \varphi_{\tau,\mathbf{x}+\hat{i}a} e^{-S_0[\varphi^*,\varphi]} \Bigg].\end{aligned} \tag{3.22}$$

The two terms in this equation are both noninteracting expectation values. The first term is the expectation value of the function $F_1(\varphi^*, \varphi) = (\varphi^*_{\tau,\mathbf{x}+\hat{i}a} \varphi_{\tau,\mathbf{x}})$ and the second term is the expectation value of the function $F_2(\varphi^*, \varphi) = (\varphi^*_{\tau,\mathbf{x}} \varphi_{\tau,\mathbf{x}+\hat{i}a})$.

Equation (2.24) will be used to compute these expectation values, but before proceeding, the following substitutions will be made to represent the indices of φ and φ^* more compactly:

$$\begin{aligned} m' &= \tau, \mathbf{x} + \hat{i}a \\ n &= \tau, \mathbf{x}, \end{aligned} \tag{3.23}$$

and in the final steps of this calculation the indices will be replaced with the originals which depend on τ and $\mathbf{x}$. Inserting the appropriate derivatives and recalling that

$$\mathbf{\Delta}[\mathbf{b}^*, \mathbf{b}] = \sum_{i,j=1}^{n} b_i^* \Delta_{i,j} b_j, \tag{3.24}$$

where $\Delta_{i,j}$ is the inverse of the matrix defining the quadratic form of the action corresponding to the noninteracting theory, S_0. The calculation then becomes

$$\langle \hat{L} \rangle_0 = -\frac{a^2}{2m} \sum_{m',n} \sum_i \left[(u_i^* - 1) \frac{\partial}{\partial b_{m'}} \frac{\partial}{\partial b_n^*} e^{\mathbf{\Delta}[\mathbf{b}^*,\mathbf{b}]} + (u_i - 1) \frac{\partial}{\partial b_n} \frac{\partial}{\partial b_{m'}^*} e^{\mathbf{\Delta}[\mathbf{b}^*,\mathbf{b}]} \right] \Bigg|_{\mathbf{b}=\mathbf{b}^*=0}, \tag{3.25}$$

which, after the derivatives have been performed and the limit $\mathbf{b} = \mathbf{b}^* = 0$ has been taken, becomes

$$\langle \hat{L} \rangle_0 = -\frac{a^2}{2m} \sum_{m',n} \sum_i \Big[(u_i^* - 1) \Delta_{n,m'} + (u_i - 1) \Delta_{m',n} \Big]. \tag{3.26}$$

The intermediate steps of this calculation can be found in Appendix B. Replacing m' and n with the original indices, this result for $\langle \hat{L} \rangle_0$ is obtained:

$$\langle \hat{L} \rangle_0 = -\frac{a^2}{2m} \sum_{\tau,\mathbf{x}} \sum_i \Big[(u_i^* - 1) \Delta_{(\tau,\mathbf{x}),(\tau,\mathbf{x}+\hat{i}a)} + (u_i - 1) \Delta_{(\tau,\mathbf{x}+\hat{i}a),(\tau,\mathbf{x})} \Big]. \tag{3.27}$$

Recalling Wick's Theorem, it is clear that the result for this expectation value is consistent with Wick's Theorem. Because there is only one derivative with respect to $\mathbf{b}^*$

and only one derivative with respect to $\mathbf{b}$, there is only one possible pair of indices that $\Delta_{i,j}$ could take. This is the simplest expectation value of the ones that $\langle \hat{L} \rangle$ depends on, but the consistency with Wick's Theorem will be seen with the more complicated expectation values calculated next.

The next expectation value to be calculated in pursuit of calculating $\langle \hat{L} \rangle$ is the noninteracting expectation value of V. This calculation is carried out using Eq. (2.24) and the equation for V (Eq. (3.10)), and again making a substitution for the sake of performing this calculation with more compact indices. The following substitution is made:

$$\begin{aligned} m &= \tau - a, \mathbf{x} \\ n &= \tau, \mathbf{x}. \end{aligned} \tag{3.28}$$

Inserting the appropriate derivatives, the calculation for $\langle V \rangle_0$ is

$$\langle V \rangle_0 = \frac{a^4}{4} \sum_{m,n} \Big(\frac{\partial^2}{\partial^2 b_m^*} \frac{\partial^2}{\partial^2 b_n} \Big) e^{\mathbf{\Delta}[\mathbf{b}^*, \mathbf{b}]} \Big|_{\mathbf{b}=\mathbf{b}^*=0}, \tag{3.29}$$

which becomes

$$\langle V \rangle_0 = \frac{a^4}{4} \sum_{m,n} 2\Delta_{m,n}^2 = \frac{a^4}{4} \sum_{\tau,\mathbf{x}} 2\Delta_{(\tau-a,\mathbf{x}),(\tau,\mathbf{x})}^2. \tag{3.30}$$

Again, the intermediate steps of this calculation are shown in Appendix B. In this calculation, there are two derivatives with respect to $\mathbf{b}^*$ and two with respect to $\mathbf{b}$. Both derivatives with respect to $\mathbf{b}^*$ take the index m and both derivatives with respect to $\mathbf{b}$ take the index n. As before, this means that there is only one possible pair of indices that can be made, namely m and n. This time, however, the result is a product of two terms as a result of the four total derivatives, which is why the $\Delta_{m,n}$ is squared.

The final noninteracting expectation value to compute is $\langle LV \rangle_0$. This is done as if LV were defined as a new function, $F_3[\varphi^*, \varphi]$. The calculation is carried out using the same methods that were used in the previous two calculations (i.e. Eq. (2.24)). The same index substitutions were made using m, m', and n as defined in Eqs. (3.23) and (3.28). It is worth noting that the derivatives with respect to b take different indices and in this calculation, more than the previous calculations, it is very important to pay attention to these indices. It is also important to note that derivatives with respect to $\mathbf{b}$ and $\mathbf{b}^*$ commute, therefore the order in which these derivatives are taken does not change the result. The expectation value takes the following form:

$$\begin{aligned} \langle \hat{L}V \rangle_0 = -\frac{a^2}{2m} \sum_{\tau,\mathbf{x}} \sum_i \frac{1}{Z_0} \Big[&(u_i^* - 1) \int \mathcal{D}\varphi^* \mathcal{D}\varphi (\varphi^*_{\tau,\mathbf{x}+\hat{i}a} \varphi_{\tau,\mathbf{x}})(\varphi^*_{\tau,\mathbf{x}} \varphi_{\tau-a,\mathbf{x}})^2 e^{-S_0[\varphi^*,\varphi]} \\ &+ (u_i - 1) \int \mathcal{D}\varphi^* \mathcal{D}\varphi (\varphi^*_{\tau,\mathbf{x}} \varphi_{\tau,\mathbf{x}+\hat{i}a})(\varphi^*_{\tau,\mathbf{x}} \varphi_{\tau-a,\mathbf{x}})^2 e^{-S_0[\varphi^*,\varphi]} \Big]. \end{aligned} \tag{3.31}$$

Inserting the appropriate derivatives into Eq. (2.24), this calculation becomes

$$\begin{aligned}\langle \hat{L}V\rangle_0 = -\frac{a^2}{2m}\sum_{m,m',n}\sum_i\Bigg[&(u_i^*-1)\left(\frac{\partial}{\partial b_{m'}}\frac{\partial}{\partial b_n^*}\right)\left(\frac{\partial}{\partial b_n}\frac{\partial}{\partial b_m^*}\right)^2 e^{\mathbf{\Delta}[\mathbf{b}^*,\mathbf{b}]}\\ &+(u_i-1)\left(\frac{\partial}{\partial b_n}\frac{\partial}{\partial b_{m'}^*}\right)\left(\frac{\partial}{\partial b_n}\frac{\partial}{\partial b_m^*}\right)^2 e^{\mathbf{\Delta}[\mathbf{b}^*,\mathbf{b}]}\Bigg]\Bigg|_{\mathbf{b}=\mathbf{b}^*=0}.\end{aligned} \tag{3.32}$$

Computing the derivatives, taking the limit $\mathbf{b} = \mathbf{b}^* = 0$, and restoring the original indices, the result is

$$\begin{aligned}\langle \hat{L}V\rangle_0 = -\frac{a^2}{2m}\sum_{\tau,\mathbf{x}}\sum_i\Big[&(u_i^*-1)(2\Delta_{(\tau,\mathbf{x}),(\tau,\mathbf{x}+\hat{i}a)}\Delta^2_{(\tau-a,\mathbf{x}),(\tau,\mathbf{x})}\\ &+4\Delta_{(\tau-a,\mathbf{x}),(\tau,\mathbf{x})}\Delta_{(\tau,\mathbf{x}),(\tau,\mathbf{x})}\Delta_{(\tau-a,\mathbf{x}),(\tau,\mathbf{x}+\hat{i}a)})\\ &+(u_i-1)(6\Delta_{(\tau,\mathbf{x}+\hat{i}a),(\tau,\mathbf{x})}\Delta^2_{(\tau-a,\mathbf{x}),(\tau,\mathbf{x})})\Big].\end{aligned} \tag{3.33}$$

The intermediate steps of this calculation can be found in appendix B. Although this calculation is more complicated because there are more indices to consider, and therefore more possible sets of pairings, the result is still consistent with Wick's Theorem.

Combining the three noninteracting expectation values calculated above [Eqs. (3.27), (3.30), and (3.33)] according to Eq. (3.15), the final equation for the interacting expectation value of the angular momentum for a complex boson field in a rotating frame is shown below:

$$\begin{aligned}\langle \hat{L}\rangle = \sum_{\tau,\mathbf{x}}\sum_i -\Bigg(&\frac{a^2}{2m}\Big[(u_i^*-1)\Delta_{(\tau,\mathbf{x}),(\tau,\mathbf{x}+\hat{i}a)}+(u_i-1)\Delta_{(\tau,\mathbf{x}+\hat{i}a),(\tau,\mathbf{x})}\Big]\\ -\lambda\Bigg\{&\frac{a^2}{2m}\Big[(u_i^*-1)(2\Delta_{(\tau,\mathbf{x}),(\tau,\mathbf{x}+\hat{i}a)}\Delta^2_{(\tau-a,\mathbf{x}),(\tau,\mathbf{x})}+4\Delta_{(\tau-a,\mathbf{x}),(\tau,\mathbf{x})}\Delta_{(\tau,\mathbf{x}),(\tau,\mathbf{x})}\Delta_{(\tau-a,\mathbf{x}),(\tau,\mathbf{x}+\hat{i}a)})\\ &+(u_i-1)(6\Delta_{(\tau,\mathbf{x}+\hat{i}a),(\tau,\mathbf{x})}\Delta^2_{(\tau-a,\mathbf{x}),(\tau,\mathbf{x})})\Big]\\ -\left(\frac{a^4}{2}\Delta^2_{(\tau-a,\mathbf{x}),(\tau,\mathbf{x})}\right)&\left(\frac{a^2}{2m}\Big[(u_i^*-1)\Delta_{(\tau,\mathbf{x}),(\tau,\mathbf{x}+\hat{i}a)}+(u_i-1)\Delta_{(\tau,\mathbf{x}+\hat{i}a),(\tau,\mathbf{x})}\Big]\right)\Bigg\}\Bigg).\end{aligned} \tag{3.34}$$

Notably, the above expression is valid in arbitrary spatial dimensions, i.e. we may use it to study rotating bosons in either 2D or 3D (or even higher dimensions if need be). It can implemented on a specific spacetime lattice and with appropriate values for the physical parameters (temperature, coupling constant, etc.), it is possible to evaluate $\langle \hat{L}\rangle$ numerically. From that answer, by studying its behavior as a function of the rotational frequency Ω, it is possible to obtain the moment of inertia (namely the derivative of $\langle \hat{L}\rangle$ with respect to Ω). It is also possible to study the temperature dependence of all of the above quantities. The numerical evaluation of this formula is left to future work, but it is possible to discuss the trends one might expect to see when examining angular frequency and temperature dependencies.

Before beginning this discussion, it is important to remember that no assumptions about the temperature of the system were made. For this reason, Eq. 3.34 should appropriately reflect expected behavior in both high and low temperature regimes. At high temperatures, Bose Einstein Condensation will not occur because it is only for temperatures *below* the critical temperature that BEC is observed. As a result, the Bose gas does not enter the superfluid phase and quantized vortices will not form when the system is rotated. The Bose gas will act like a normal fluid would when subjected to rotation: it will undergo rigid body rotation where, for frequencies below the critical frequency, the angular momentum will have a linear dependence on the angular frequency. This trend is shown in Figure 3.6 below.

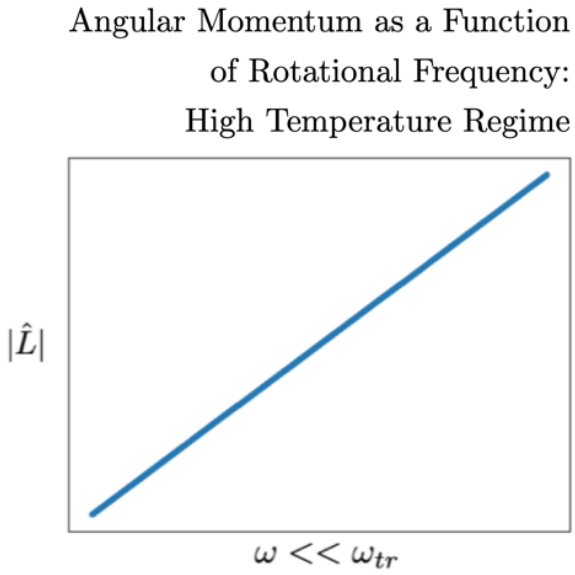

FIGURE 3.6: Rotational frequency dependence of angular momentum at high temperature.

Notice that the x-axis only spans frequencies that are much smaller than the trapping frequency. This is because as the frequency of rotation approaches the trapping frequency, the most energetic particles will have sufficient energy to escape the trapped system. This is equivalent to how water will begin to escape the bucket if it is rotated too fast. We expect that there will be a linear relationship between the angular momentum and the angular frequency for sufficiently small frequencies, but that as the angular frequency approaches the trapping frequency, the dependence will become non-linear but still continuous.

At low temperatures, the Bose gas will experience BEC and quantized vortices will form when the system is coupled to rotation. The mere formation of the quantized vortices means that the Bose gas does not experience rigid body rotation. Instead, discontinuous jumps in the angular momentum should appear as the angular frequency increases and the number of quantized vortices increases. This expected trend is shown in Figure 3.7 below.

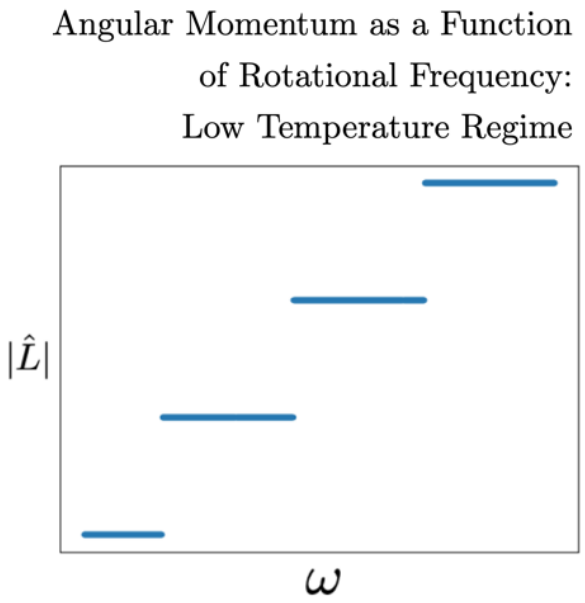

FIGURE 3.7: Rotational frequency dependence of angular momentum at low temperature.

The figure above is a simplified version of the trend we would expect to see. In reality, the discontinuous jumps will not be as clearly defined because of the fact that the formation of vortices are non-perturbative, meaning our perturbative approximation of $\langle \hat{L} \rangle$ could have difficulty detecting this phenomenon. Our approximation should demonstrate the correct behavior in the high temperature regime and in the low temperature regime, there should be some kind of behavior indicating jumps in the angular momentum, even if they are subtle. The approximation presented in Eq. 3.34 should still provide numerical support for full Monte-Carlo calculations even if some trends are not discussed above can not be clearly observed.

Chapter 4

Summary and Conclusion

Ultracold atoms are a fascinating platform to study quantum many-body physics. Experimentalists have come a long way since the first realization of bosonic and fermionic condensates and the field continues to grow as it explores these systems under a wide range of conditions.

In this work we have focused on one of those situations, namely rotating bosonic systems at finite temperature. This is a challenging area of study because, already in the absence of rotation, the inter-atomic interactions make the many-body problem very difficult to treat with paper-and-pencil methods. Typically, including those interactions would involve some kind of Monte Carlo simulation. Here, we have taken steps toward making contact with such future simulations by calculating the relevant expressions for the angular momentum in lattice perturbation theory.

To that end, we have presented the relevant elements of quantum statistical mechanics and outlined their connection with the path integral formulation of quantum field theory. The latter was instrumental in deriving the perturbative expansion of the partition function and general observables, which are accessed by way of Wick's theorem. This powerful theorem connects expectation values of arbitrary operators in an interacting theory with combinations of (known) single-particle propagators in the non-interacting theory.

After presenting the main aspects of the formalism, we showed the lattice action and isolated the coupling to angular momentum, which on the lattice is effected through so-called link variables, as in lattice gauge theories. That enabled us to take the next step and calculate the form of the expectation value of the angular momentum at first order in perturbation theory, which was our final result. Because the determination of the final form of this expectation value did not involve assumptions about the system itself, namely the temperature of the system, we expect the numerical results to follow, at least somewhat, expected trends in both the high and low temperature regimes.

Future work will include the explicit evaluation of the expressions obtained by specifying the physical as well as lattice parameters, and eventually specifying a renormalization procedure (i.e. a matching connecting the theory to a physical process) to fix the value of the coupling constant. From the thermodynamic behavior of the angular momentum, a few thermodynamic response functions can be obtained, such as the moment of inertia and the temperature response of rotation. Using the expressions derived, and the explicit dependence of the propagator Δ on the thermodynamic parameters, those response functions can be obtained. These are, however, left for future studies as well.

Appendix A

General Gaussian Integrals with Complex Variables

Consider the following multi-dimensional, general Gaussian integral with complex variables $\mathbf{z}$ and $\mathbf{z}^*$:

$$Z(\mathbf{A}, \mathbf{b}, \mathbf{b}^*) = \int d^n z \, d^n z^* \exp[-A(\mathbf{z}, \mathbf{z}^*) + \mathbf{b} \cdot \mathbf{z}^* + \mathbf{b}^* \cdot \mathbf{z}], \tag{A.1}$$

where $\mathbf{b} \cdot \mathbf{z}^*$ and $\mathbf{b}^* \cdot \mathbf{z}$ are inner products as defined by equation 2.16 and $A(\mathbf{z}, \mathbf{z}^*)$ takes the quadratic form:

$$A(\mathbf{z}, \mathbf{z}^*) = \sum_{i,j=1}^{n} z_i^* A_{i,j} z_j. \tag{A.2}$$

Equation (A.1) can be reduced to some form of the the following integral,

$$Z(\mathbf{A}) = \int d^n z \, d^n z^* \exp[-\sum_{i,j=1}^{n} w_i^* M_{i,j} w_j] = (2\pi)^n \det(\mathbf{A})^{-1}, \tag{A.3}$$

the solution of which is well known. Notice that the only difference between the solution above and the solution in Eq. (2.13) is the factor of $1/2$ in the exponents.

The goal, then, is to rewrite the exponent in Eq. (A.1) as $\mathbf{w}^* \mathbf{A} \mathbf{w}$, where $\mathbf{w}$ and $\mathbf{w}^*$ are new variables that depend on $\mathbf{z}$ and $\mathbf{z}^*$, plus some term(s) that do not depend on $\mathbf{z}$ or $\mathbf{z}^*$. This will be done by completing the square. For the sake of simplicity, these calculations will be done without the use of indices. The terms in the exponent are

$$(\mathbf{z}^* \cdot \mathbf{A} \cdot \mathbf{z}) - (\mathbf{z}^* \cdot \mathbf{b}) - (\mathbf{z} \cdot \mathbf{b}^*). \tag{A.4}$$

Inserting the matrix $\mathbf{A}$ and its inverse $\mathbf{A}^{-1}$ in the latter two terms, the exponent becomes

$$\begin{aligned} &(\mathbf{z}^* \cdot \mathbf{A} \cdot \mathbf{z}) - (\mathbf{z}^* \cdot \mathbf{A} \cdot \mathbf{A}^{-1} \cdot \mathbf{b}) - (\mathbf{b}^* \cdot \mathbf{A}^{-1} \cdot \mathbf{A} \cdot \mathbf{z}) \\ &\quad = (\mathbf{z}^* - \mathbf{b}^* \cdot \mathbf{A}^{-1}) \cdot \mathbf{A} \cdot \mathbf{z} - (\mathbf{z}^* \cdot \mathbf{A} \cdot \mathbf{A}^{-1} \cdot \mathbf{b}). \end{aligned} \tag{A.5}$$

In order to "collect like terms" again, one must add and subtract $(\mathbf{z}^* - \mathbf{b}^* \cdot \mathbf{A}^{-1}) \cdot \mathbf{b}$. Equation (A.5) then becomes

$$\begin{aligned} (\mathbf{z}^* - \mathbf{b}^* \cdot \mathbf{A}^{-1}) \cdot \mathbf{A} \cdot \mathbf{z} - (\mathbf{z}^* \cdot \mathbf{A} \cdot \mathbf{A}^{-1} \cdot \mathbf{b}) - (\mathbf{z}^* - \mathbf{b}^* \cdot \mathbf{A}^{-1}) \cdot \mathbf{A} \cdot \mathbf{A}^{-1} \cdot \mathbf{b} \\ + (\mathbf{z}^* - \mathbf{b}^* \cdot \mathbf{A}^{-1}) \cdot \mathbf{A} \cdot \mathbf{A}^{-1} \cdot \mathbf{b}, \end{aligned} \tag{A.6}$$

which reduces to

$$= (\mathbf{z}^* - \mathbf{b}^* \cdot \mathbf{A}^{-1}) \cdot \mathbf{A} \cdot (\mathbf{z} - \mathbf{A}^{-1} \cdot \mathbf{b}) - \mathbf{b}^* \cdot \mathbf{A}^{-1} \cdot \mathbf{b}. \tag{A.7}$$

Inserting this equivalent form of the exponent into Eq. (A.1), the integral becomes

$$Z(\mathbf{A}, \mathbf{b}, \mathbf{b}^*) = \int d^n z \, d^n z^* \exp[-(\mathbf{z}^* - \mathbf{b}^* \cdot \mathbf{A}^{-1}) \cdot \mathbf{A} \cdot (\mathbf{z} - \mathbf{A}^{-1} \cdot \mathbf{b}) + \mathbf{b}^* \cdot \mathbf{A}^{-1} \cdot \mathbf{b}], \tag{A.8}$$

which can be written as

$$= \int d^n w \, d^n w^* \exp[-\mathbf{w}^* \cdot \mathbf{A} \cdot \mathbf{w}] \exp[\mathbf{b}^* \cdot \mathbf{A}^{-1} \cdot \mathbf{b}], \tag{A.9}$$

where $\mathbf{w}^* = (\mathbf{z}^* - \mathbf{b}^* \cdot \mathbf{A}^{-1})$ and $\mathbf{w} = (\mathbf{z} - \mathbf{A}^{-1} \cdot \mathbf{b})$.

Because $\mathbf{A}$, $\mathbf{b}$, and $\mathbf{b}^*$ are all independent of $\mathbf{w}$ and $\mathbf{w}^*$, the second exponential term can be pulled outside of the integral. The remaining integral is of the form presented in Eq. (A.3), meaning the solution of this integral can be written down:

$$Z(\mathbf{A}, \mathbf{b}, \mathbf{b}^*) = (2\pi)^n \det(\mathbf{A})^{-1} e^{\mathbf{\Delta}(\mathbf{b}, \mathbf{b}^*)}, \tag{A.10}$$

with

$$\mathbf{\Delta}(\mathbf{b}, \mathbf{b}^*) = \mathbf{b}^* \cdot \mathbf{\Delta} \cdot \mathbf{b} = \sum_{i,j=1}^{n} b_i^* \Delta_{i,j} b_j, \tag{A.11}$$

and where $\mathbf{\Delta} = \mathbf{A}^{-1}$ and $\Delta_{i,j} = A_{i,j}^{-1}$.

Appendix B

Line-by-Line Calculations for Chapter 3

Complete calculation for $\langle \hat{L} \rangle_0$:

$$\langle \hat{L} \rangle_0 = -\frac{a^6}{8m} \sum_{m',n} \sum_i \left[(u_i^\dagger - \mathbb{1}) \frac{\partial}{\partial b_{m'}} \frac{\partial}{\partial b_n^*} e^{\mathbf{\Delta}[\mathbf{b}^*,\mathbf{b}]} + (u_i - \mathbb{1}) \frac{\partial}{\partial b_n} \frac{\partial}{\partial b_{m'}^*} e^{\mathbf{\Delta}[\mathbf{b}^*,\mathbf{b}]} \right] \Big|_{\mathbf{b}=\mathbf{b}^*=0} \tag{B.1}$$

$$\begin{aligned} \langle \hat{L} \rangle_0 = -\frac{a^6}{8m} \sum_{m',n} \sum_i \Big[& (u_i^\dagger - \mathbb{1}) \frac{\partial}{\partial b_{m'}} \Big(\sum_j \Delta_{n,j} b_j \Big) e^{\mathbf{\Delta}[\mathbf{b}^*,\mathbf{b}]} \\ & + (u_i - \mathbb{1}) \frac{\partial}{\partial b_n} \Big(\sum_j \Delta_{m',j} b_j \Big) e^{\mathbf{\Delta}[\mathbf{b}^*,\mathbf{b}]} \Big] \Big|_{\mathbf{b}=\mathbf{b}^*=0} \end{aligned} \tag{B.2}$$

$$\langle \hat{L} \rangle_0 = -\frac{a^6}{8m} \sum_{m',n} \sum_i \Big[(u_i^\dagger - 1) \Delta_{n,m'} + (u_i - 1) \Delta_{m',n} \Big]. \tag{B.3}$$

Complete calculation for $\langle V \rangle_0$:

$$\langle V \rangle_0 = \frac{a^4}{4} \sum_{m,n} \Big(\frac{\partial^2}{\partial^2 b_m^*} \frac{\partial^2}{\partial^2 b_n} \Big) e^{\mathbf{\Delta}[\mathbf{b}^*,\mathbf{b}]} \Big|_{\mathbf{b}=\mathbf{b}^*=0}, \tag{B.4}$$

$$\langle V \rangle_0 = \frac{a^4}{4} \sum_{m,n} \Big(\frac{\partial^2}{\partial^2 b_m^*} \frac{\partial}{\partial b_n} \Big) \Big(\sum_i b_i^* \Delta_{i,n} \Big) e^{\mathbf{\Delta}[\mathbf{b}^*,\mathbf{b}]} \Big|_{\mathbf{b}=\mathbf{b}^*=0} \tag{B.5}$$

$$\langle V \rangle_0 = \frac{a^4}{4} \sum_{m,n} \Big(\frac{\partial^2}{\partial^2 b_m^*} \Big) \Big(\sum_i b_i^* \Delta_{i,n} \Big)^2 e^{\mathbf{\Delta}[\mathbf{b}^*,\mathbf{b}]} \Big|_{\mathbf{b}=\mathbf{b}^*=0} \tag{B.6}$$

$$\begin{aligned} \langle V \rangle_0 = \frac{a^4}{4} \sum_{m,n} \Big(\frac{\partial}{\partial b_m^*} \Big) \Big[& \Big(\sum_i b_i^* \Delta_{i,n} \Big)^2 \Big(\sum_j \Delta_{m,j} b_j \Big) e^{\mathbf{\Delta}[\mathbf{b}^*,\mathbf{b}]} \Big] \\ & + 2\Delta_{m,n} \Big(\sum_i b_i^* \Delta_{i,n} \Big) e^{\mathbf{\Delta}[\mathbf{b}^*,\mathbf{b}]} \Big] \Big|_{\mathbf{b}=\mathbf{b}^*=0} \end{aligned} \tag{B.7}$$

$$\langle V \rangle_0 = \frac{a^4}{4} \sum_{m,n} 2\Delta_{m,n}^2 \tag{B.8}$$

Complete calculation for $\langle \hat{L}V \rangle_0$:

$$\begin{aligned}\langle \hat{L}V \rangle_0 = -\frac{a^6}{8m} \sum_{m,m',n} \sum_i \Bigg[(u_i^\dagger - \mathbb{1}) \left(\frac{\partial}{\partial b_{m'}} \frac{\partial}{\partial b_n^*} \right) \left(\frac{\partial}{\partial b_n} \frac{\partial}{\partial b_m^*} \right)^2 e^{\mathbf{\Delta}[\mathbf{b}^*,\mathbf{b}]} \\ + (u_i - \mathbb{1}) \left(\frac{\partial}{\partial b_n} \frac{\partial}{\partial b_{m'}^*} \right) \left(\frac{\partial}{\partial b_n} \frac{\partial}{\partial b_m^*} \right)^2 e^{\mathbf{\Delta}[\mathbf{b}^*,\mathbf{b}]} \Bigg] \Bigg|_{\mathbf{b}=\mathbf{b}^*=0}.\end{aligned} \tag{B.9}$$

$$\begin{aligned}\langle V \rangle_0 = -\frac{a^6}{8m} \sum_{m,m',n} \sum_i \Bigg[(u_i^\dagger - \mathbb{1}) \left(\frac{\partial}{\partial b_{m'}} \frac{\partial}{\partial b_n^*} \right) \left(\frac{\partial^2}{\partial^2 b_m^*} \frac{\partial}{\partial b_n} \right) \Big(\sum_i b_i^* \Delta_{i,n} \Big) e^{\mathbf{\Delta}[\mathbf{b}^*,\mathbf{b}]} \\ + (u_i - \mathbb{1}) \left(\frac{\partial}{\partial b_n} \frac{\partial}{\partial b_{m'}^*} \right) \left(\frac{\partial^2}{\partial^2 b_m^*} \frac{\partial}{\partial b_n} \right) \Big(\sum_i b_i^* \Delta_{i,n} \Big) e^{\mathbf{\Delta}[\mathbf{b}^*,\mathbf{b}]} \Bigg] \Bigg|_{\mathbf{b}=\mathbf{b}^*=0}\end{aligned} \tag{B.10}$$

$$\begin{aligned}\langle V \rangle_0 = -\frac{a^6}{8m} \sum_{m,m',n} \sum_i \Bigg[(u_i^\dagger - \mathbb{1}) \left(\frac{\partial}{\partial b_{m'}} \frac{\partial}{\partial b_n^*} \right) \left(\frac{\partial^2}{\partial^2 b_m^*} \right) \Big(\sum_i b_i^* \Delta_{i,n} \Big)^2 e^{\mathbf{\Delta}[\mathbf{b}^*,\mathbf{b}]} \\ + (u_i - \mathbb{1}) \left(\frac{\partial}{\partial b_n} \frac{\partial}{\partial b_{m'}^*} \right) \left(\frac{\partial^2}{\partial^2 b_m^*} \right) \Big(\sum_i b_i^* \Delta_{i,n} \Big)^2 e^{\mathbf{\Delta}[\mathbf{b}^*,\mathbf{b}]} \Bigg] \Bigg|_{\mathbf{b}=\mathbf{b}^*=0}\end{aligned} \tag{B.11}$$

$$\begin{aligned}\langle V \rangle_0 = -\frac{a^6}{8m} \sum_{m,m',n} \sum_i \Bigg[(u_i^\dagger - \mathbb{1}) \left(\frac{\partial}{\partial b_{m'}} \frac{\partial}{\partial b_n^*} \right) \left(\frac{\partial}{\partial b_m^*} \right) \Bigg[2\Delta_{m,n} \Big(\sum_i b_i^* \Delta_{i,n} \Big) e^{\mathbf{\Delta}[\mathbf{b}^*,\mathbf{b}]} \\ + \Big(\sum_i b_i^* \Delta_{i,n} \Big)^2 \Big(\sum_j \Delta_{m,j} b_j \Big) e^{\mathbf{\Delta}[\mathbf{b}^*,\mathbf{b}]} \Bigg] \\ + (u_i - \mathbb{1}) \left(\frac{\partial}{\partial b_n} \frac{\partial}{\partial b_{m'}^*} \right) \left(\frac{\partial}{\partial b_m^*} \right) \Bigg[2\Delta_{m,n} \Big(\sum_i b_i^* \Delta_{i,n} \Big) e^{\mathbf{\Delta}[\mathbf{b}^*,\mathbf{b}]} \\ + \Big(\sum_i b_i^* \Delta_{i,n} \Big)^2 \Big(\sum_j \Delta_{m,j} b_j \Big) e^{\mathbf{\Delta}[\mathbf{b}^*,\mathbf{b}]} \Bigg] \Bigg] \Bigg|_{\mathbf{b}=\mathbf{b}^*=0}\end{aligned} \tag{B.12}$$

$$\begin{aligned}\langle \hat{L}V\rangle_0 = -\frac{a^6}{8m}\sum_{m,m',n}\sum_i\Bigg[(u_i^\dagger - \mathbb{1})\left(\frac{\partial}{\partial b_{m'}}\frac{\partial}{\partial b_n^*}\right)\Bigg[2\Delta_{m,n}^2 e^{\mathbf{\Delta}[\mathbf{b}^*,\mathbf{b}]} \\ +4\Delta_{m,n}\Big(\sum_i b_i^*\Delta_{i,n}\Big)\Big(\sum_j \Delta_{m,j}b_j\Big)e^{\mathbf{\Delta}[\mathbf{b}^*,\mathbf{b}]} \\ +\Big(\sum_i b_i^*\Delta_{i,n}\Big)^2\Big(\sum_j \Delta_{m,j}b_j\Big)^2 e^{\mathbf{\Delta}[\mathbf{b}^*,\mathbf{b}]}\Bigg] \\ +(u_i - \mathbb{1})\left(\frac{\partial}{\partial b_n}\frac{\partial}{\partial b_{m'}^*}\right)\Bigg[2\Delta_{m,n}^2 e^{\mathbf{\Delta}[\mathbf{b}^*,\mathbf{b}]} \\ +4\Delta_{m,n}\Big(\sum_i b_i^*\Delta_{i,n}\Big)\Big(\sum_j \Delta_{m,j}b_j\Big)e^{\mathbf{\Delta}[\mathbf{b}^*,\mathbf{b}]} \\ +\Big(\sum_i b_i^*\Delta_{i,n}\Big)^2\Big(\sum_j \Delta_{m,j}b_j\Big)^2 e^{\mathbf{\Delta}[\mathbf{b}^*,\mathbf{b}]}\Bigg]\Bigg]\Bigg|_{\mathbf{b}=\mathbf{b}^*=0}\end{aligned} \tag{B.13}$$

$$\begin{aligned}\langle \hat{L}V\rangle_0 = -\frac{a^6}{8m}\sum_{m,m',n}\sum_i\Bigg[(u_i^\dagger - \mathbb{1})\left(\frac{\partial}{\partial b_{m'}}\right)\Bigg[2\Delta_{m,n}^2\Big(\sum_j \Delta_{n,j}b_j\Big)e^{\mathbf{\Delta}[\mathbf{b}^*,\mathbf{b}]} \\ +4\Delta_{m,n}\Delta_{n,n}\Big(\sum_j \Delta_{m,j}b_j\Big)e^{\mathbf{\Delta}[\mathbf{b}^*,\mathbf{b}]} \\ +4\Delta_{m,n}\Big(\sum_i b_i^*\Delta_{i,n}\Big)\Big(\sum_j \Delta_{m,j}b_j\Big)\Big(\sum_j \Delta_{n,j}b_j\Big)e^{\mathbf{\Delta}[\mathbf{b}^*,\mathbf{b}]} \\ +2\Delta_{n,n}\Big(\sum_i b_i^*\Delta_{i,n}\Big)\Big(\sum_j \Delta_{m,j}b_j\Big)^2 e^{\mathbf{\Delta}[\mathbf{b}^*,\mathbf{b}]} \\ +\Big(\sum_i b_i^*\Delta_{i,n}\Big)^2\Big(\sum_j \Delta_{m,j}b_j\Big)^2\Big(\sum_j \Delta_{n,j}b_j\Big)e^{\mathbf{\Delta}[\mathbf{b}^*,\mathbf{b}]}\Bigg] \\ +(u_i - \mathbb{1})\left(\frac{\partial}{\partial b_n}\right)\Bigg[2\Delta_{m,n}^2\Big(\sum_j \Delta_{m',j}b_j\Big)e^{\mathbf{\Delta}[\mathbf{b}^*,\mathbf{b}]} \\ +4\Delta_{m,n}\Delta_{m',n}\Big(\sum_j \Delta_{m,j}b_j\Big)e^{\mathbf{\Delta}[\mathbf{b}^*,\mathbf{b}]} \\ +4\Delta_{m,n}\Big(\sum_i b_i^*\Delta_{i,n}\Big)\Big(\sum_j \Delta_{m,j}b_j\Big)\Big(\sum_j \Delta_{m',j}b_j\Big)e^{\mathbf{\Delta}[\mathbf{b}^*,\mathbf{b}]} \\ +2\Delta_{m'n}\Big(\sum_i b_i^*\Delta_{i,n}\Big)\Big(\sum_j \Delta_{m,j}b_j\Big)^2 e^{\mathbf{\Delta}[\mathbf{b}^*,\mathbf{b}]} \\ +\Big(\sum_i b_i^*\Delta_{i,n}\Big)^2\Big(\sum_j \Delta_{m,j}b_j\Big)^2\Big(\sum_j \Delta_{m',j}b_j\Big)e^{\mathbf{\Delta}[\mathbf{b}^*,\mathbf{b}]}\Bigg]\Bigg]\Bigg|_{\mathbf{b}=\mathbf{b}^*=0}\end{aligned} \tag{B.14}$$

$$\langle \hat{L}V \rangle_0 = -\frac{a^6}{8m} \sum_{m,m',n} \sum_i \Big[(u_i^\dagger - \mathbb{1}) \big[(2\Delta_{n,m'} \Delta_{m,n}^2 + 4\Delta_{m,n}\Delta_{n,n}\Delta_{m,m'}) \big] + (u_i - \mathbb{1})(6\Delta_{m',n}\Delta_{m,n}^2) \Big] \tag{B.15}$$

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